Final Exam Mathematical Physics, Prof. G. Palasantzas

- Date 17-06-2016
- Total number of points 100
- 10 points free for coming to the final exam
- For all problems justify your answer

<u>Problem 1</u> (10 points)

We form the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, and we show that it is covergent using the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1 \quad \text{with} \quad x \in (-\infty, +\infty)$$

Therefore, since the series is convergent we have: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$

Problem 2 (15 points)

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This is a subcase of the more general series $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b>0$ with b=3 and a=5. The

solution is (assuming below that b=3 and a=5):

a $a_n = \frac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\frac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b \Leftrightarrow -b < x-a < b \Leftrightarrow$ $a-b < x < a+b$. When $|x-a| = b$, $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

if we substitute b=3 and a=5 we have range of convergence **I=(2, 8)**

Problem 3 (20 points)

CASE III $c^2 - 4mk < 0$ **(underdamping)**
Here the roots are complex:

If we solve the auxiliary equation we have two complex roots:

$$r_{1,2} = -(c/2m) \pm j\omega\sqrt{1-(c/2m\omega)^2} \quad (k = m\omega^2) \Rightarrow$$

$$\text{Homogenous solution}: x_{homo}(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right]$$

$$\text{with } \tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$$

(see book chap. 17.3 & Nestor site)

We look for a particular solution of the form: $x_p(t) = \text{A}\cos(\omega t) + \text{B}\sin(\omega t)$

Look in Nestor the general solution of the AFM-equation of motion with $\omega=\omega_o$:

since $\omega = \omega_o$ (and as a result $k - m\omega^2 = 0$) we obtain after substition into the equation of motion: $c\omega\text{B} = \text{F}_o$ and $\text{A} = 0$

$$\Rightarrow x_p(t) = \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

Total solution: $x(t) = x_{homo}(t) + x_p(t)$

$$x(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

Problem 4 (10 points)

Upon substitution in the differential equation we obtain

$$\sum_{n=-\infty}^{+\infty} (-n^2 + A) W_n e^{jnx} = \sum_{n=-\infty}^{+\infty} f_n e^{jnx} \Rightarrow$$

$$(-n^2 + A) W_n = f_n \Rightarrow$$

$$W_n = \frac{f_n}{A - n^2} \quad (A \neq n^2)$$

therefore we have $\boxed{W(x) = \sum_{n=-\infty}^{+\infty} \frac{f_n}{A - n^2} e^{jnx}}$

Problem 5 (15 points)

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(a) $\mathcal{F}_x[\sin(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\left(\frac{e^{2\pi i k_0 x} - e^{-2\pi i k_0 x}}{2i}\right) dx$

$= \frac{1}{2}\, i \int_{-\infty}^{\infty} \left[-e^{-2\pi i (k-k_0)x} + e^{-2\pi i (k+k_0)x}\right] dx = \frac{1}{2}\, i\, [\delta(k+k_0) - \delta(k-k_0)],$

(b) since *sin(3x)=3sin(x)-4sin*^{3}*(x) we obtain sin*^{3}*(x)=[3sin(x)-sin(3x)]/4. As a result the Fourier transform sin*^{3}*(x) will be*

$\mathcal{F}_x[\mathrm{Sin}^3(2\pi k_0 x)](k) = [\,3\,\mathcal{F}_x[\mathrm{Sin}(2\pi k_0 x)](k) - \mathcal{F}_x[\mathrm{Sin}(6\pi k_0 x)](k)\,] / 4$

Since

$\mathcal{F}_x[\mathrm{Sin}(2\pi k_0 x)](k) = (i/2)[\delta(k+k_0) - \delta(k-k_0)]$

$\mathcal{F}_x[\mathrm{Sin}(6\pi k_0 x)](k) = (i/2)[\delta(k+3k_0) - \delta(k-3k_0)]$

We obtain $\mathcal{F}_x[\mathrm{Sin}^3(2\pi k_0 x)](k) = (3i/8)[\delta(k+k_0) - \delta(k-k_0)] - (i/8)[\delta(k+3k_0) - \delta(k-3k_0)]$

<u>You can also calculate (b) directly as in (a).</u>

Problem 6 (20 points)

(a)

The solution is determined by the separation of variables (the Fourier method):

$$u(x,t) = F(x)G(t).$$

Then

$$\frac{\partial u}{\partial t} = FG', \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

Substituting this into one-dimensional heat equation and separating variables,

$$FG' = c^2 F''G$$

$$\frac{G'}{c^2 G} = \frac{F''}{F} = const = -p^2$$

we obtain the differential equations for $G(t)$ and $F(x)$

$$G' + c^2 p^2 G = 0,$$

$$F'' + p^2 F = 0.$$

Satisfy the boundary conditions:

$$u(0,t) = F(0)G(t) = 0, \qquad u(L,t) = F(L)G(t) = 0, \quad t \geq 0.$$

Thus,

$$F(0) = 0, \qquad F(L) = 0.$$

The general solution for F is

$$F = A\cos px + B\sin px.$$

and

$$F(0) = 0: \qquad A = 0; \qquad F(L) = 0: \qquad B\sin pL = 0$$

which yields

$$\sin pL = 0 \quad (B \neq 0)$$

$$pL = n\pi, \quad p = p_n = \frac{n\pi}{L} \quad (n = 1, 2, \ldots).$$

$$F = F_n = \sin p_n x = \sin\frac{n\pi}{L}x \quad (n = 1, 2, \ldots).$$

The equation for G becomes

$$G' + \lambda_n^2 G = 0, \quad \lambda_n = \frac{cn\pi}{L}.$$

The general solution of this equation is

$$G(t) = G_n(t) = B_n e^{-\lambda_n^2 t} \quad (n = 1, 2, \ldots).$$

Hence the solutions of

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

satisfying

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t \geq 0.$$

are

$$u_n(x,t) = F_n(x)G_n(t) = B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots).$$

These functions are called **eigenfunctions** and

$$\lambda_n = \frac{cn\pi}{L}$$

are called **eigenvalues**.

Now we can solve the entire problem by setting

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x.$$

Satisfy the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x).$$

Thus,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx, \qquad n = 1, 2, \ldots.$$

(b) Only the m-term of the solution u(x,t) is non-zero, so that we obtain B_m=A and the solution has the form $u(x,t)=A\exp[-(cm\pi/L)^2 t] \sin(m\pi x/L)$

Then if you substitute m=9 you obtain $\mathbf{u(x,t)=A\exp[-(c9\pi/L)^2 t] \sin(9\pi x/L)}$

(c) Only the m_1, m_2, m_3- terms of the solution u(x,t) are non-zero, so that we obtain Bm_1=A1, Bm_2=A2, Bm_3=A3 and the solution has the form

$u(x,t) = A1 \exp[-(cm_1\pi/L)^2 t] \sin(m_1\pi x/L) + A2 \exp[-(cm_2\pi/L)^2 t] \sin(m_2\pi x/L)+$

$A2 \exp[-(cm_3\pi/L)^2 t] \sin(m_3\pi x/L)$,

Therefore after substitution of $(m_1, m_2, m_3)=(10, 20, 30)$ we obtain:

$u(x,t) = A1 \exp[-(c10\pi/L)^2 t] \sin(10\pi x/L) + A2 \exp[-(c20\pi/L)^2 t] \sin(20\pi x/L) + A2 \exp[-(c30\pi/L)^2 t] \sin(30\pi x/L)$